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FORMULATION OF BOUNDARY-VALUE PROBLEMS FOR THE DYNAMICS OF TWO-DIMENSIONAL SYSTEMS WITH MOVING LOADS AND FASTENINGS†

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A load or fastening is moving in a two-dimensional system without separation. The interdependent dynamical behaviour of these two elements is investigated. The Hamilton variational principle is used to formulate a self-consistent boundary-value problem which correctly incorporates the forces of interaction in the moving contact, including those due to the relative motion and wave pressure. The equations of energy and momentum transport are derived. It is shown that the intermediary through which the vibrational energy of the two-dimensional system is converted into the kinetic energy of the one-dimensional object is the wave pressure force. As an example, a boundary-value problem is formulated for the motion of a beam along a Kirchhoff model plate.

1. CONSIDER a mechanical system consisting of a two-dimensional elastic strip, along which a one-dimensional load or fastening is moving without separation (Fig. 1). Throughout this paper a "one-dimensional load" will mean a system, possessing elastic and inertial properties, which is described satisfactorily by a one-dimensional model such as a string, beam or the like. The nature of the vibrations of the two-dimensional system will depend on the law of motion of the load; on the other hand, the motion of the latter is affected both by impressed forces and by the reaction forces exerted by the two-dimensional system. Our problem is thus to describe the coordinated motion of the two systems.

Let x and y be the spatial variables of the two-dimensional system, t the time, $D=\{(x,y,t)$: $x_1\leqslant x\leqslant x_2,\, y_1\leqslant y\leqslant y_2,\, t_1\leqslant t\leqslant t_2\}$ a domain in xyt space and $D_0=\{(y,t)$: $y_1\leqslant y\leqslant y_2,\, t_1\leqslant t\leqslant t_2\}$ the projection of D on to the yt plane. We shall assume that the law governing the motion of the one-dimensional load is represented by some generalized coordinate $l(y,t)$ and a collection of vector-valued functions of the generalized coordinates $\mathbf{v}(y,t)$ and $\mathbf{w}(y,t)$, both of dimensions n, such that

$$l(y,t)\in C^2(D_0),\quad v^k(y,t)\in C^2(D_0)$$

$$w^k(y,t)\in C^2(D_0),\quad (k=1,\ldots,n)$$

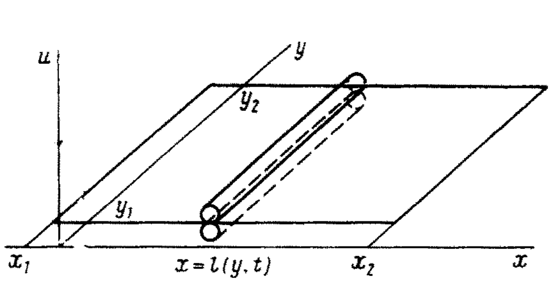

FIG. 1.

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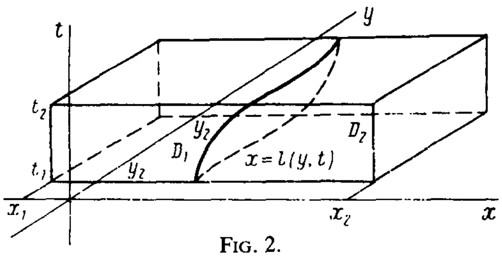

FIG. 2.

The surface $x = l(y, t)$, $(y, t) \in D_0$ divides D into parts D_1 and D_2 (Fig. 2). The law governing the motion of the two-dimensional system is represented by another vector-valued function of the generalized coordinates:

$$\mathbf{u}(x, y, t) = (u^1(x, y, t), \ldots, u^n(x, y, t))$$

$$(u^k(x, y, t) \in C^1(D),\ u^k(x, y, t) \in C^1(D_i),\ i = 1, 2,\ k = 1, \ldots, n)$$

Let $\lambda(x, y, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_t, \mathbf{u}_{xx}, \mathbf{u}_{xy}, \mathbf{u}_{yy})$ be the density of the Lagrangian of the two-dimensional system and

$$L(y, t, l, l_y, l_t, l_{yy}, \mathbf{v}, \mathbf{v}_y, \mathbf{v}_t, \mathbf{v}_{yy}, \mathbf{w}, \mathbf{w}_y, \mathbf{w}_t, \mathbf{w}_{yy})$$

the density of the Lagrangian of the one-dimensional system; λ and L are assumed to be continuously differentiable in all arguments. The function λ may have the form

$$\lambda = \begin{cases} \lambda_1, & x < l(y, t) \\ \lambda_2, & x > l(y, t) \end{cases}$$

where λ_1 and λ_2 are twice continuously differentiable functions in all their arguments. When we say that the objects are moving without separation, we mean that the following equalities hold:

$$v^k(y, t) = u^k(l(y, t), y, t)$$

$$w^k(y, t) = u_x{}^k(l(y, t), y, t),\quad (y, t) \in D_0,\ k = 1, \ldots, n$$

A sequence of functions $[l(y, t), \mathbf{u}(x, y, t), \mathbf{v}(y, t), \mathbf{w}(y, t)]$ that satisfy the above conditions will be called strongly consistent.†

The following theorem is proved by the standard methods of variational calculus, using Hamilton's principle [1].

Theorem. In order for a strongly consistent sequence $[l(y, t), \mathbf{u}(x, y, t), \mathbf{v}(y, t), \mathbf{w}(y, t)]$ to determine a stationary value of the functional

$$J = \sum_{i=1}^{2} \iiint_{D_i} \lambda\, dx\, dy\, dt + \iint_{D_0} L\, dy\, dt$$

the functions $\mathbf{u}(x, y, t)$, $l(y, t)$, $\mathbf{v}(y, t)$, $\mathbf{w}(y, t)$ must satisfy the equations

$$\lambda_{u^k} - \frac{\partial}{\partial t}\lambda_{u_t{}^k} - \frac{\partial}{\partial x}\lambda_{u_x{}^k} - \frac{\partial}{\partial y}\lambda_{u_y{}^k} + \frac{\partial^2}{\partial x^2}\lambda_{u^k_{xx}} +$$

$$+ \frac{\partial^2}{\partial x \partial y}\lambda_{u^k_{xy}} + \frac{\partial^2}{\partial y^2}\lambda_{u^k_{yy}} = -q^k,\quad k = 1, \ldots, n,\quad (x, y, t) \in D_i,\quad i = 1, 2 \tag{1.1}$$

†VESNITSKII A. I., KAPLAN L. E., KRYSOV S. V. and UTKIN G. A., Self-consistent problems of the dynamics of one-dimensional systems with moving loads and fastenings. Preprint No. 159, Gor'kii Scientific Research Institute of Radiophysics (1982).

$$L_l - \frac{\partial}{\partial t} L_{l_t} - \frac{\partial}{\partial y} L_{l_y} + \frac{\partial^2}{\partial y^2} L_{l_{yy}} = [F] - q_l, \quad (y,t) \in D_0 \tag{1.2}$$

$$L_{v^k} - \frac{\partial}{\partial t} L_{v_t^k} - \frac{\partial}{\partial y} L_{v_y^k} + \frac{\partial^2}{\partial y^2} L_{v_{yy}^k} = [N_k^1] - q_v^k, \quad (y,t) \in D_0$$

$$L_{w^k} - \frac{\partial}{\partial t} L_{w_t^k} - \frac{\partial}{\partial y} L_{w_y^k} + \frac{\partial^2}{\partial y^2} L_{w_{yy}^k} = [N_k^2] - q_w^k, \quad (y,t) \in D_0$$

$$\begin{gathered} v^k(y,t) = u^k(l(y,t), y, t) \\ w^k(y,t) = u_x^k(l(y,t), y, t) \quad k = 1, \ldots, n, \ (y,t) \in D_0 \end{gathered} \tag{1.3}$$

Here

$$[A(x,y,t)] = A(l(y,t)+0, y, t) - A(l(y,t)-0, y, t) \tag{1.4}$$

$$F = \lambda - \sum_{k=1}^{n} (u_x^k N_k^1 + u_{xx}^k N_k^2)$$

$$N_k^1 = \lambda_{u_x^k} - \frac{\partial}{\partial x} \lambda_{u_{xx}^k} - \frac{\partial}{\partial y} \lambda_{u_{xy}^k} - l_y \left(\lambda_{u_y^k} - 2 \frac{\partial}{\partial y} \lambda_{u_{yy}^k} \right) +$$

$$+ l_{yy} \lambda_{u_{yy}^k} + l_y^2 \frac{\partial}{\partial x} \lambda_{u_{yy}^k} - l_t \lambda_{u_t^k}$$

$$N_k^2 = \lambda_{u_{xx}^k} - l_y \lambda_{u_{xy}^k} + l_y^2 \lambda_{u_{yy}^k}$$

The added functions on the right side of the equations are the densities of the generalized forces q^k, q_l, q_v^k, q_w^k, which are intrinsically non-conservative.

The differential equations (1.1) describe the dynamics of the two-dimensional system; Eqs (1.2) and (1.3) are the boundary conditions of the consistent motion, with Eqs (1.2) at the same time describing the dynamics of the non-dimensional object.

To complete the formulation of the problem, we must also add conditions at the edges of the objects and initial conditions.

2. The results may be interpreted as follows. It can be shown [1] that $p_k = \lambda_{u_t^k}$ is the density of the generalized momentum corresponding to the generalized coordinate $u^k(x,y,t)$ of the two-dimensional system, $\mathbf{T}_k$ is the vector of the internal conservative force density and M_k is the tensor of the momentum density, where

$$\mathbf{T}_k = \left\| \begin{matrix} T_{1k} \\ T_{2k} \end{matrix} \right\| = \left\| \begin{matrix} \lambda_{u_x^k} - \frac{\partial}{\partial x} \lambda_{u_{xx}^k} - \frac{1}{2} \frac{\partial}{\partial y} \lambda_{u_{xy}^k} \\ \lambda_{u_y^k} - \frac{1}{2} \frac{\partial}{\partial x} \lambda_{u_{xy}^k} - \frac{\partial}{\partial y} \lambda_{u_{yy}^k} \end{matrix} \right\|$$

$$M_k = (\mathbf{M}_{1k}, \mathbf{M}_{2k}) = \left\| \begin{matrix} \lambda_{u_{xx}^k} & \frac{1}{2} \lambda_{u_{xy}^k} \\ \frac{1}{2} \lambda_{u_{xy}^k} & \lambda_{u_{yy}^k} \end{matrix} \right\|$$

and hence, for each generalized coordinate $u^k(x,y,t)$ the law of motion (1.1) becomes

$$\partial p_k / \partial t + \operatorname{div} \mathbf{T}_k = \lambda_{u^k} + q^k$$

i.e. as usual, it can be interpreted as an equation of generalized momentum transport.

A special feature of distributed systems is intrinsic transport of energy and wave momentum [3]. To derive the transport equation, we take the scalar product of the equations of dynamics of the two-dimensional system (1.1) and the partial derivatives of the vector of generalized coordinates $\mathbf{u}_t$, $\mathbf{u}_x$, $\mathbf{u}_y$, and transform the resulting expressions to the following form:

$$
\begin{gathered}
\partial h/\partial t + \operatorname{div}\mathbf{S} = -\lambda_t + \textstyle\sum q^k u_t^k,\ \partial \mathbf{p}^*/\partial t + \operatorname{div} T^* = \mathbf{F}_0 - \textstyle\sum q^k \nabla u^k \\
h = \textstyle\sum u_t^k p_k - \lambda,\ \mathbf{S} = \textstyle\sum (u_t^k \mathbf{T}_k + (\nabla u_t^k, M_k)) \\
\mathbf{p}^* = -\textstyle\sum p_k \nabla u^k,\ \mathbf{F}_0 = \nabla\lambda = \{\lambda_x, \lambda_y\} \\
T^* = \{\mathbf{T}_1^*, \mathbf{T}_2^*\} = \begin{Vmatrix} \lambda - \Sigma_{1x} & -\Sigma_{1y} \\ -\Sigma_{2x} & \lambda - \Sigma_{2y} \end{Vmatrix} \\
\textstyle\sum_{iz} = \sum (u_z^k T_{ik} + (\nabla u_z^k, \mathbf{M}_{ik})),\ i = 1, 2,\ z = x, y
\end{gathered}
\tag{2.1}
$$

Here h is the density of the Hamiltonian (generalized energy), $\mathbf{S}$ is the energy flux density vector, $\mathbf{p}^*$ is the wave momentum density vector, T^* is the wave momentum flux density tensor (the stress tensor) and $\mathbf{F}_0$ is the vector of the recoil forces density due to the distributed reflection of waves propagating in the inhomogeneous elastic system; throughout, summation is from $k = 1$ to $k = n$.

A similar derivation yields the transport equations for the generalized energy and momentum in the one-dimensional system. Let $p_{1k} = L_{v_t^k}$, $p_{2k} = L_{w_t^k}$, $p_0 = L_{l_t}$ be the densities of the generalized momenta corresponding to the generalized coordinates $v^k(y, t)$, $w^k(y, t)$ $(k = 1, \ldots, n)$, $l(y, t)$ of the one-dimensional system and $T_{1k} = L_{v_y^k} - \partial L_{v_{yy}^k}/\partial y$, $T_{2k} = L_{w_y^k} - \partial L_{w_{yy}^k}/\partial y$, $T_0 = L_{l_y} - \partial L_{l_{yy}}/\partial y$ the internal conservative forces in a cross-section y. Then Eqs (1.2) can be rewritten as laws for the variation of the generalized momenta:

$$
\begin{gathered}
\partial p_{1k}/\partial t + \partial T_{1k}/\partial y = L_{vk} + q_v^k - [(\mathbf{n}_1, \mathbf{T}_k) - (\nabla (\mathbf{n}_1, \mathbf{M}_{2k}), \mathbf{n}_2) - l_t p_k] \\
\partial p_{2k}/\partial t + \partial T_{2k}/\partial y = L_{wk} + q_w^k - [\mathbf{n}_1 M_k \mathbf{n}_1^T] \\
\partial p_0/\partial t + \partial T_0/\partial y = L_l + q_l - [(\mathbf{n}_1, \mathbf{T}_1^*) - l_t p_1^* + (\nabla \{\textstyle\sum u_x^k (\mathbf{n}_1, \mathbf{M}_{2k})\}, \\
\mathbf{n}_2)],\ \mathbf{n}_1 = (1, -l_y),\ \mathbf{n}_2 = \begin{pmatrix} l_y \\ 1 \end{pmatrix}
\end{gathered}
$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are vectors in the x, y plane collinear with the normal and tangent vectors to the intersection of the surface $x = l(y, t)$ and a plane $t = \text{const}$.

To obtain the laws governing energy and wave momentum transport in the one-dimensional system, we multiply Eqs (1.2) by the appropriate first partial derivatives of the generalized coordinates $\mathbf{v}(y, t)$, $\mathbf{w}(y, t)$, $l(y, t)$; the resulting equations can be reduced to the form

$$
\begin{gathered}
\partial h_0/\partial t + \partial S_0/\partial y = -L_t + l_t q_l + \textstyle\sum (v_t^k q_v^k + w_t^k q_w^k) - [(\mathbf{n}_1, \mathbf{S}) - \\
- l_t h - (\nabla \{\textstyle\sum u_t^k (\mathbf{n}_1, \mathbf{M}_{2k})\}, \mathbf{n}_2\}] \\
\partial p_0^*/\partial t + \partial T_0^*/\partial y = L_y - (l_y q_l + \textstyle\sum (v_y^k q_v^k + w_y^k q_w^k)) - [(\mathbf{n}_1, \mathbf{T}_2^*) - \\
- l_t p_2^* + (\nabla \{\textstyle\sum u_y^k (\mathbf{n}_1, \mathbf{M}_{2k})\}, \mathbf{n}_2)] \\
h_0 = l_t p_0 + \textstyle\sum (v_t^k p_{1k} + w_t^k p_{2k}) - L \\
S_0 = l_t T_0 + l_{yy} L_{l_{yy}} + \textstyle\sum (v_t^k T_{1k} + w_t^k T_{2k} + v_{yt}^k L_{v_{yy}^k} + w_{yt}^k L_{w_{yy}^k}) \\
p_0^* = -l_y p_0 - \textstyle\sum (v_y^k p_{1k} + w_y^k p_{2k}) \\
T_0^* = L - (l_y T_0 + l_{yy} L_{l_{yy}}) - \textstyle\sum (v_y^k T_{1k} + w_y^k T_{2k} + v_{yy}^k L_{v_{yy}^k} + w_{yy}^k L_{w_{yy}^k})
\end{gathered}
$$

Here h_0 is the density of the Hamiltonian, S_0 is the energy flux density, p_0^* is the wave momentum density and T_0^* is the wave momentum flux density.

Besides the local laws of transport, we are also interested in the global laws governing the variations of energy and wave momentum. Integrating Eqs (2.1) over the domain $D^* = \{(x, y)$: $x_1 \leqslant x \leqslant x_2,\ y_1 \leqslant y \leqslant y_2\}$, we obtain

$$
\frac{dH}{dt} = \iint_{D^*} \Big(\sum u_t^k q^k - \lambda_t\Big)\, dx\, dy - \oint_{\Gamma} (\mathbf{n}, \mathbf{S})\, dl + \int_{y_1}^{y_2} [(\mathbf{n}_1, \mathbf{S}) - l_t h]\, dy \tag{2.2}
$$

$$\frac{d\mathbf{P}^*}{dt} = \iint_{D^*} \left(\mathbf{F}_0 - \sum q^k \nabla u^k\right) dx\,dy - \oint_{\Gamma} (\mathbf{n}, T^*)\,dl + \int_{y_1}^{y_2} [(\mathbf{n}_1, T^*) - l_t \mathbf{p}^*]\,dy$$

$$H(t) = \iint_{D^*} h(x, y, t)\,dx\,dy, \qquad \mathbf{P}^*(t) = \iint_{D^*} \mathbf{p}^*(x, y, t)\,dx\,dy$$

where Γ is the boundary of D^*, $\mathbf{n}$ is the vector of the outward normal to the latter and $\mathbf{n}_1 = (1, -l_y)$.

Thus, for a system with constant parameters ($\lambda_t = 0$), when there are no non-conservative forces ($q^k = 0$), the change in total energy $H(t)$ is due to the energy flux S through the boundaries Γ and $x = l(y, t)$. For example, if the elastic strip is absolutely rigidly fastened ($u = 0$, $\partial u/\partial \mathbf{n} = 0$) at the boundaries $y = y_1$, $y = y_2$, $x = x_1$, $x = l(t)$, Eq (2.2) may be written, using (1.4) in the form

$$\frac{dH(t)}{dt} = -l^{\cdot}(t) F_g, \qquad F_g = \int_{y_1}^{y_2} F(l(t) - 0, y, t)\,dy$$

where F_g is the wave pressure force on the moving fastening, i.e. the wave pressure force becomes the intermediary whereby vibrational energy is converted into energy of translational motion and vice versa.

3. As an example, let us formulate the problem of flexural vibrations $u(x, y, t)$ of a Kirchhoff plate [4] with

$$\lambda = {}^1/_2 (\rho u_t^2 - D\{(u_{xx} + u_{yy})^2 + 2(1 - \nu)(u_{xy}^2 - u_{xx} u_{yy})\})$$

$$D = Eh^3/(12(1 - \nu))$$

where a beam is moving along the plate without separation:

$$L = {}^1/_2 (\rho_0 (u_t^{0\,2} + l_t^2 + J_0 \varphi_t^{0\,2}) - E_0 J (u_{yy}^{0\,2} + l_{yy}^2) - G_0 J_0 \varphi_y^{0\,2} - k_0 u^{0\,2})$$

performing flexural vibrations $u^0(y, t)$, $l(y, t)$ and torsional vibrations $\varphi^0(y, t)$. Here D is the cylindrical stiffness, E is Young's modulus, ν is Poisson's ratio, ρ is the surface density, h is the thickness of the plate, E_0 and G_0 are Young's modulus and the shear modulus, ρ_0 is the density per unit length, J_0 and J are the polar moment of inertia and moment of inertia of the cross-section relative to axes perpendicular to the beam axis and K_0 is the coefficient of the elastic bed. The flexural vibrations of the plate are determined by solving Eq. (1.1):

$$\rho u_{tt} + D(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) = q$$

and finding a solution that satisfies the conditions of continuity and smoothness (1.3) at $x = l(y, t)$:

$$u^0(y, t) = u(l(y, t) - 0, y, t) = u(l(y, t) + 0, y, t)$$

$$\varphi^0(y, t) = u_x(l(y, t) - 0, y, t) = u_x(l(y, t) + 0, y, t)$$

The equations of balance for the flexural torques and transversal forces (1.2) are

$$\rho_0 J_0 \varphi_{tt}^0 - G_0 J_0 \varphi_{yy}^0 = [N^2] + q_\varphi$$

$$\rho_0 u_{tt}^0 + E_0 J u_{yyyy}^0 + k_0 u^0 = [N^1] + q_u$$

Here

$$N^2 = D(u_{xx} + \nu u_{yy} - 2(1 - \nu) l_y u_{xy} + l_y^2 (u_{yy} + \nu u_{xx}))$$

$$N^1 = \rho l_t u_t + D(-u_{xxx} - (2 - \nu) u_{xyy} + l_y (\nu u_{xxy} + u_{yyy}) + l_{yy} (\nu u_{xx} + u_{yy}) + l_y^2 (\nu u_{xxx} + u_{xyy}))$$

The equation of motion of the beam $l(y, t)$ is

$$\rho_0 l_{tt} + E_0 J l_{yyyy} = [F] + q_l$$

$$F = -\lambda - u_x N^1 - u_{xx} N^2$$

where F is the wave pressure force.

Several other examples of specific boundary-value problems may be found in [5].

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